

A Quasi Curtis-Tits-Phan theorem for the symplectic group

Rieuwert J. Blok and Corneliu Hoffman
 Department of Mathematics and Statistics
 Bowling Green State University
 Bowling Green, OH 43403-1874

February 2, 2008

Key Words: symplectic group, amalgam, opposite

AMS subject classification (2000): Primary 51A50; Secondary 57M07.

Abstract

We obtain the symplectic group as an amalgam of low rank subgroups akin to Levi components. We do this by having the group act flag-transitively on a new type of geometry and applying Tits' lemma. This provides a new way of recognizing the symplectic groups from a small collection of small subgroups.

1 Introduction

In the revision of the classification of finite simple groups one of the important steps requires one to prove that if a simple group G (the minimal counterexample) contains a certain amalgam of subgroups that one normally finds in a known simple group H then G is isomorphic to H . A geometric approach to recognition theorems was initiated in [1, 2, 7]. The present paper provides a new recognition theorem for symplectic groups. It is a natural generalization of the Curtis-Tits-Phan theory from above.

Let us describe the general geometric approach. We consider a group G which is either semi-simple of Lie type or a Kac-Moody group. Let $\mathcal{T} = (B_+, B_-)$ be the associated twin building. We first define a *flip* to be an involutory automorphism σ of \mathcal{T} that interchanges the two halves, preserves distances and codistances and takes at least one chamber to an opposite.

Given a flip σ , construct \mathcal{C}_σ as the chamber system whose chambers are the pairs of opposite chambers (c, c^σ) of \mathcal{T} . Let G_σ be the fixed subgroup under the σ -induced automorphism of G . We refer to [1] for details on the construction. Whenever the geometry Γ_σ is simply connected one obtains G_σ as the universal completion of the amalgam of maximal parabolics for the action of G_σ on Γ_σ .

Consider, the building of type A_n associated to $G = \mathrm{PSL}_{n+1}(\mathbb{F})$ for some field \mathbb{F} . The objects of type $i \in \{1, 2, \dots, n\}$ are the i -spaces of a fixed $(n+1)$ -dimensional vector space V over \mathbb{F} . Two objects X and Y are opposite if $\langle X, Y \rangle = V$ and $X \cap Y$ is minimal.

Let σ be a flip. Then it is immediate that σ is induced by a polarity. The requirement that σ -invariant pairs of opposite chambers exist enforces that at least one 1-space p does not intersect its polar hyperplane: that is, p is not absolute. These polarities are well-known and classified to correspond to orthogonal or unitary forms. The objects of the geometry Γ_σ are the non-degenerate subspaces with respect to the polarity. Since a symplectic polarity has no non-absolute points, the A_n building does not have a symplectic flip. The aim of this paper is to deal with this exceptional situation. We shall construct a geometry Γ similar to Γ_σ and obtain a presentation of the symplectic group. This geometry will have higher rank than the usual building geometry of the symplectic group and, in particular, will provide an amalgam presentation for $\mathrm{Sp}_4(q)$, a group that falls out of the scope of Curtis-Tits-Phan program.

Let V be a vector space of dimension $2n \geq 4$ over a field \mathbb{F} endowed with a symplectic form \mathbf{s} of maximal rank. Let $I = \{1, 2, \dots, n\}$. Moreover consider $\mathcal{H} = \{e_i, f_i\}_{i \in I}$ a hyperbolic basis of V . The group of linear automorphisms of V preserving the form \mathbf{s} will be called the *symplectic group of V* and is denoted $G = \mathrm{Sp}(V)$. We shall define a geometry Γ on the subspaces of V whose radical has dimension at most 1. This geometry is transversal, residually connected, and simply connected. Moreover, G acts flag-transitively on Γ . This then gives a presentation of G as the universal completion of the amalgam \mathcal{A} of its maximal parabolic subgroups with respect to its action on Γ . Induction allows us to replace the amalgam \mathcal{A} of maximal parabolic subgroups by the amalgam $\mathcal{A}_{\leq 2}$ of parabolic subgroups of rank at most 2. A refinement of the amalgam $\mathcal{A}_{\leq 2}$ then leads to the following setup.

Consider the following amalgam $\mathcal{A}^\pi = \{M_i^\pi, S_j^\pi, M_{ik}^\pi, S_{jl}^\pi, Q_{ij}^\pi\}_{j,l \in I; i,k \in I - \{n\}}$, whose groups are characterized as follows:

- S_i^π is the stabilizer in G of all elements of $\mathcal{H} - \{e_i, f_i\}$ and the subspace $\langle e_i, f_i \rangle$,
- M_i^π is the stabilizer in G of all elements of $\mathcal{H} - \{f_i, f_{i+1}\}$ and the subspace $\langle e_i, f_i, e_{i+1}, f_{i+1} \rangle$,
- $M_{ij}^\pi = \langle M_i^\pi, M_j^\pi \rangle$, $S_{ij}^\pi = \langle S_i^\pi, S_j^\pi \rangle$, $Q_{ij}^\pi = \langle M_i^\pi, S_j^\pi \rangle$.

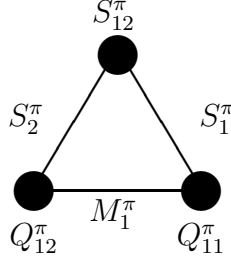
We describe these groups in more detail in Section 6. In particular, in Lemma 6.8 we show that the groups in this amalgam are very small. For instance,

$$\begin{aligned} S_i^\pi &\cong \mathrm{Sp}_2(\mathbb{F}), \\ M_i^\pi &\cong \mathbb{F}^3. \end{aligned}$$

Our main result is the following.

Theorem 1.1 *If $|\mathbb{F}| > 2$ then the symplectic group $\mathrm{Sp}(V)$ is the universal completion of the amalgam \mathcal{A}^π . Moreover for any field \mathbb{F} the symplectic group is the universal completion of the amalgam \mathcal{A} from Section 5.2*

Corollary 1.2 *The group $\mathrm{Sp}_4(\mathbb{F})$ is the universal completion of the following amalgam. $\{S_1^\pi, S_2^\pi, M_1^\pi, S_{12}^\pi, Q_{11}^\pi, Q_{12}^\pi\}$. Here, $S_{12}^\pi \cong \mathrm{Sp}_2(\mathbb{F}) \times \mathrm{Sp}_2(\mathbb{F})$ and $Q_{11}^\pi \cong Q_{12}^\pi \cong \mathbb{F}^3 \rtimes \mathrm{Sp}_2(\mathbb{F})$.*



The paper is organized as follows. In Section 2 we review some basic notions on geometries and some relevant facts about symplectic spaces. In Section 3 we introduce a geometry Γ on the almost non-degenerate subspaces of V with respect to some symplectic form of maximal rank and describe its residues. We prove that this geometry is transversal and residually connected. In Section 4 we show that the geometry and all residues of rank at least 3 are simply connected with one exception. We show that in the exceptional case the residue has a simply connected 2-cover. In Section 5 we describe the flag-transitive action of $\mathrm{Sp}(V)$ on Γ and its rank 3 residues. We describe the parabolic subgroups in some detail and prove that $\mathrm{Sp}(V)$ is the universal completion of the amalgam $\mathcal{A}_{\leq 2}$ of parabolics of rank at most 2 for its action on Γ . In Section 6 we define a slim version of the amalgam $\mathcal{A}_{\leq 2}$ by removing most of the Borel subgroup from each of its groups. Finally, in Section 7 we define the amalgam \mathcal{A}^π and prove Theorem 1.1.

2 Preliminaries

We first need some notation to describe how \mathbf{s} restricts to the various subspaces of V . Let \perp denote the orthogonality relation between subspaces of V induced by \mathbf{s} . Thus, for $U, W \leq V$ we have

$$U \perp W \iff \mathbf{s}(u, w) = 0 \text{ for all } u \in U, w \in W.$$

We write

$$U^\perp = \{v \in V \mid \mathbf{s}(u, v) = 0 \ \forall u \in U\} \leq V.$$

The *radical* of a subspace U is the subspace $\text{Rad}(U) = U \cap U^\perp$. The *rank* of U is $\text{rank}(U) = \dim U - \dim \text{Rad}(U)$. Note that since \mathbf{s} is symplectic, we have $\mathbf{s}(v, v) = 0$ for all $v \in V$ and so V has no anisotropic part with respect to \mathbf{s} .

Let $2r = \text{rank}(U)$ and $d = \dim(U) - 2r$. A *hyperbolic basis* for U is a basis $\{e_i, f_j \mid 1 \leq i \leq r + d, 1 \leq j \leq r\}$ such that

(i) for all $1 \leq i, j \leq r$,

$$\begin{aligned} \mathbf{s}(e_i, e_j) = \mathbf{s}(f_i, f_j) &= 0, \\ \mathbf{s}(e_i, f_j) &= \delta_{ij}, \text{ and} \end{aligned}$$

(ii) $\{e_{r+i} \mid 1 \leq i \leq d\}$ is a basis for $\text{Rad}(U)$.

Lemma 2.1 *Suppose that $W \leq U - \text{Rad}(U)$. Then, any hyperbolic basis for W extends to a hyperbolic basis for U .*

Proof This is in some sense Witt's theorem. □

Lemma 2.2 *Suppose that $W \leq U - \text{Rad}(U)$ and $r = \text{rank}(U)$. If $\dim(W) = 2r$, then $U = W \oplus \text{Rad}(U)$ and W is non-degenerate.*

Proof That $U = W \oplus \text{Rad}(U)$ is simple linear algebra. As a consequence, and since $\text{Rad}(U) \perp U$, we have $\text{Rad}(W) \leq \text{Rad}(U)$. Thus $\text{Rad}(W) \leq \text{Rad}(U) \cap W = \{0\}$. □

We now introduce some basic notions on geometries as we view them.

Definition 2.3 A *pre-geometry* is a triple $\Gamma = (\mathcal{O}, \text{typ}, I, \star)$, where \mathcal{O} is a collection of objects, I is a set of *types*, \star is a symmetric and reflexive relation, called the *incidence relation* and $\text{typ}: \mathcal{O} \rightarrow I$ is a surjective *type function* such that whenever $X \star Y$, then either $X = Y$ or $\text{typ}(X) \neq \text{typ}(Y)$. The *rank* of the geometry Γ is the size of I .

Definition 2.4 The *incidence graph* of Γ is the graph whose vertices are the objects

of Γ and in which two objects are adjacent if they are incident in Γ . This is a multipartite graph whose parts are indexed by I .

Definition 2.5 A *flag* F is a collection of pairwise incident objects. The *rank* of F is $\text{rank}(F) = |F|$. The *type* of F is $\text{typ}(F) = \{\text{typ}(X) \mid X \in F\}$. A *chamber* is a flag C of type I .

A *residue* R of a flag F of type $I - J$ is the pre-geometry induced on the collection of all objects incident to F , but not belonging to F . We say that R has type J .

Definition 2.6 Given subsets $J, K \subseteq I$ and an $(I - J)$ -flag F . Then the *K-shadow* of F is the collection of all K -flags incident to F .

Definition 2.7 A pre-geometry Γ is *transversal* if any flag is contained in a chamber.

Definition 2.8 A pre-geometry is *connected* if the incidence graph is connected. A pre-geometry is *residually connected* if each of its residues of rank at least 2 is connected.

Definition 2.9 A *geometry* is a pre-geometry that is transversal, connected and residually connected.

Definition 2.10 An *automorphism group* G of a pre-geometry Γ is a group of permutations of the collection of objects that preserves type and incidence. We call G *flag-transitive* if for any $J \subseteq I$, G is transitive on the collection of J -flags.

Definition 2.11 Let G be a group of automorphisms of a geometry Γ over an index set I . Fix a maximal flag C . The *standard parabolic subgroup of type $J \subseteq I$* is the stabilizer in G of the residue of type J on C .

Let us recall the definition of the fundamental group of a connected geometry Δ . A *path of length k* is a sequence of elements x_0, \dots, x_k such that x_i and x_{i+1} are incident for $0 \leq i < k$. We do not allow repetitions, that is, $x_i \neq x_{i+1}$ for all $0 \leq i < k$. A *cycle based at an element x* is a path x_0, \dots, x_k in which $x_0 = x = x_k$. Two paths γ and δ are *homotopy equivalent* if one can be obtained from the other by inserting or eliminating cycles of length 2 or 3. We denote this by $\gamma \simeq \delta$. The equivalence classes of cycles based at an element x form a group under concatenation. This group is called the *fundamental group of Δ based at x* and is denoted $\pi_1(\Delta, x)$. If Δ is (path) connected, then the isomorphism type of this group does not depend on x and we call this group simply the *fundamental group* of Δ and denote it $\pi_1(\Delta)$.

We call Δ *simply connected* if $\pi_1(\Delta)$ is trivial.

The central tool for this paper is the following result by J. Tits.

Lemma 2.12 *Given a group G acting flag-transitively on a geometry Γ . Fix a maximal flag C . Then G is the universal completion of the amalgam consisting of the standard maximal parabolics with respect to C if and only if Γ is simply connected.*

In order to prove that Δ is simply connected, it suffices to show that any cycle based at a given element x is homotopy equivalent to a cycle of length 0. We call such a cycle *trivial* or *null homotopic*.

Lemma 2.13 *If Δ is a pre-geometry, then any cycle all of whose elements are incident to a given element is null-homotopic.*

Proof Let $\gamma = (x_0, \dots, x_k)$ be a cycle all of whose elements are incident to a given element y . Then

$$\begin{aligned} \gamma &\simeq (x_0, y) \circ (y, x_0, x_1, y) \circ (y, x_1, x_2, y) \circ \dots \circ (y, x_{k-1}, x_k, y) \circ (y, x_0) \\ &\simeq (x_0, y) \circ (y, x_1, x_2, y) \circ \dots \circ (y, x_{k-1}, x_k, y) \circ (y, x_0) \\ &\simeq (x_0, y) \circ (y, x_0) \\ &\simeq 0. \end{aligned}$$

□

All our pre-geometries will have a string diagram. We will give the following ad-hoc definition here.

Definition 2.14 We say that a pre-geometry *has a string diagram* if there is a total ordering on its type set I such that for any three types $i, j, k \in I$ with $i < j < k$ we have the following. If X, Y, Z are objects of type i, j , and k respectively such that X and Z are incident with Y , then X is incident with Z .

Note that if a pre-geometry has a string diagram, then so does every residue.

For a geometry with string diagram, by *points* we mean the objects whose type p is minimal in I and by *lines* we mean the objects whose type is minimal in $I - \{p\}$.

We now prove the following result, which is also a consequence of Theorem 12.64 in Pasini [10].

Lemma 2.15 *Let Δ be a geometry with a string diagram. Then, every cycle based at x is homotopy equivalent to a cycle consisting of points and lines only.*

Proof Consider an arbitrary cycle $\gamma = x_0, \dots, x_k$ based at x . By transversality, γ is homotopic to a cycle based at the point incident to x . We may therefore assume that x itself is a point.

We prove that γ is homotopic to a cycle of points and lines only by induction on m , the number of elements other than points and lines on γ . Clearly if $m = 0$ we are done.

Let i be minimal such that x_i is not a point or a line. If $\text{typ}(x_{i+1}) > \text{typ}(x_i)$, then γ is homotopy equivalent to $\gamma' = x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k$, since $x_{i-1}, x_i, x_{i+1}, x_{i-1}$ is a 3-cycle. Clearly γ' contains $m - 1$ elements that are not points or lines, so by induction we are done.

Now assume $\text{typ}(x_{i+1}) < \text{typ}(x_i)$. Note that x_i is not a point or a line. Let y be a point incident to x_{i+1} . It exists by transversality. Note that y is incident to x_i as well.

By residual connectedness of the residue of x_i , there is a path δ of points and lines on x_i connecting x_{i-1} to y . Then, x_{i-1}, x_i, x_{i+1} is homotopic to $\delta \circ (y, x_{i+1})$ by Lemma 2.13. Hence γ is homotopic to $\gamma' = (x_0, \dots, x_{i-1}) \circ \delta \circ (y, x_{i+1}, x_{i+2}, \dots, x_k)$. Again, γ' contains $m - 1$ elements that are not points or lines, so by induction we are done. \square

Definition 2.16 Given a geometry Γ with type set I and $k \in \mathbb{N}_{\geq 1}$, a k -cover is a geometry $\bar{\Gamma}$ with type set I together with an incidence and type preserving map $\pi: \bar{\Gamma} \rightarrow \Gamma$ with the following properties:

- For any J -flag F in Γ the fiber $\pi^{-1}(F)$ consists of exactly k distinct and disjoint J -flags.
- Given a flag F in Γ and some flag \bar{F} in $\bar{\Gamma}$ such that $\pi(\bar{F}) = F$, then the restriction $\pi: \text{Res}(\bar{F}) \rightarrow \text{Res}(F)$ is an isomorphism.

We employ the following ad-hoc definition. We call $(\bar{\Gamma}, \pi)$ the *universal cover* of Γ if it is a cover of Γ that is simply connected.

The fundamental group $\Pi_x(\Gamma)$ based at the point x acts on $\pi^{-1}(x)$ as follows. As we know $\Pi_x(\Gamma)$ is generated by cycles that are not null-homotopic. Let $\delta = x = x_0, x_1, \dots, x_k = x_0$ be such a cycle. Pick a point $\bar{x} \in \pi^{-1}(x)$. By the covering properties, there is a unique object of type $\text{typ}(x_1)$ incident to \bar{x} . Call this object \bar{x}_1 . Continuing in this way, we find a path $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_k$. Note that \bar{x}_0 and \bar{x}_k are (not necessarily equal) elements in $\pi^{-1}(x)$. Set $\delta \cdot \bar{x}_0 = \bar{x}_k$.

The following is a special case of Lemma 12.2 of [10].

Lemma 2.17 *Let Γ be a geometry with universal cover $(\bar{\Gamma}, \pi)$ and let x be an object. Then the fundamental group $\Pi_x(\Gamma)$ acts regularly on $\pi^{-1}(x)$.*

In particular, the universal cover of Γ is a 2-cover if and only if the fundamental group of Γ is $\mathbb{Z}/2\mathbb{Z}$.

3 A geometry for the symplectic group

Let V be a vector space of dimension n over a field \mathbb{F} endowed with a symplectic form \mathbf{s} of maximal rank. The *quasi-Phan geometry* $\Gamma = \Gamma(V)$ is defined as follows. For $i \in I = \{1, 2, \dots, n-1\}$, the i -objects, or objects of type i , are the i -spaces $U \leq V - \text{Rad}(V)$ such that $\dim(\text{Rad}(U)) \leq 1$. More explicitly, since \mathbf{s} is symplectic, this means that

$$\dim(\text{Rad}(U)) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

We say that two objects X and Y are *incident* whenever $X \subseteq Y - \text{Rad}(Y)$ or vice versa.

It is not too difficult to see that $\text{Sp}(V)$ is an automorphism group for Γ . In fact, Lemma 5.1 shows that it acts flag-transitively on Γ .

Corollary 3.1 *Any hyperbolic basis for V gives rise to a unique chamber of Γ and, conversely, any chamber gives rise to a (not necessarily unique) hyperbolic basis for V .*

Proof Let C be a chamber. Then, for any two consecutive objects $W, U \in C \cup \{V\}$ we have $W \leq U - \text{Rad}(U)$. Hence, using Lemma 2.1 repeatedly we find a hyperbolic basis \mathcal{H} for V such that, for any $X \in C$, $\mathcal{H} \cap X$ is a hyperbolic basis for X .

Conversely, let \mathcal{H} be a hyperbolic basis for V . Set $d = \dim(\text{Rad}(V))$. We have $\mathcal{H} = \{e_i, f_j \mid 1 \leq i \leq r+d, 1 \leq j \leq r\}$. For $1 \leq i \leq r+d$, let $h_{2i-1} = e_i$ and, for $1 \leq j \leq r$, let $h_{2j} = f_j$. Then, setting $C_l = \langle h_1, h_2, \dots, h_l \rangle_V$, the collection $C = \{C_l\}_{l=1}^{n-1}$ is a chamber of Γ . \square

In the remainder of this paper we shall use the following standard setup. Let $2r = \text{rank}(V)$ and let $d = \dim(\text{Rad}(V)) \leq 1$, so that $n = 2r+d$. We fix a hyperbolic basis

$$\mathcal{H} = \{e_i, f_j \mid 1 \leq i \leq r+d, 1 \leq j \leq r\}.$$

Alternatively we write

$$\begin{aligned} \mathcal{H} &= \{h_k\}_{k=1}^n, \text{ where} \\ h_k &= \begin{cases} e_i & \text{if } k = 2i-1, \\ f_j & \text{if } k = 2j. \end{cases} \end{aligned}$$

We call this the *standard hyperbolic basis*. The *standard chamber* is the chamber $C = \{C_k\}_{k=1}^{n-1}$ associated to \mathcal{H} as in Corollary 3.1. That is, $C_k = \langle h_1, h_2, \dots, h_k \rangle_V$, for all $1 \leq k \leq n-1$.

Lemma 3.2 *The pre-geometry Γ is transversal and has a string diagram.*

Proof Let F be a flag. Then, for any two consecutive objects $W, U \in (F \cup \{V\})$ we have $W \leq U - \text{Rad}(U)$. Hence, using Lemma 2.1 repeatedly we find a hyperbolic basis \mathcal{H} for V such that, for any $X \in C$, $\mathcal{H} \cap X$ is a hyperbolic basis for X . According to Corollary 3.1, \mathcal{H} defines a unique chamber C . One verifies that $F \subseteq C$. The natural ordering on I provides Γ with a string diagram. \square

Lemma 3.3 *The pre-geometry Γ is connected. More precisely, the $\{1, 2\}$ -shadow geometry has diameter at most 2 with equality if $n \geq 3$.*

Proof Let X, Z be 1-spaces in $V - \text{Rad}(V)$. If X and Z span a non-degenerate 2-space, then we are done. In particular, if $n = 2$, then the diameter is 1.

Other wise let W be a point on $\langle X, Z \rangle$ different from X and Z . In case $\langle X, Z \rangle \supseteq \text{Rad}(V)$, let $W = \text{Rad}(V)$. Since $V/\text{Rad}(V)$ is non-degenerate, there is a point Y in W^\perp that is not in $\langle X, Z \rangle^\perp$. Then clearly $X, Z \not\subseteq Y^\perp$ and so X, Y, Z is a path in Γ from X to Z . Thus the $\{1, 2\}$ -shadow geometry has diameter at most 2. Clearly equality holds. \square

3.1 Residual geometries

Let $C = \{C_i\}_{i \in I}$ be the standard chamber of Γ associated to the hyperbolic basis \mathcal{H} . For every $J \subseteq I$ the *standard residue of type J* , denoted R_J , is the residue of the $(I - J)$ -flag $\{C_i\}_{i \in I - J}$. Let $\biguplus_{m=1}^M J_m$ be the partition of J into maximal contiguous subsets. (We call $K \subseteq I$ contiguous if, whenever $i, k \in K$ and $i < j < k$, then $j \in K$.)

In this case, $R_J = R_{J_1} \times R_{J_2} \times \cdots \times R_{J_M}$ since Γ has a string diagram. It now suffices to describe R_J , where J is contiguous. Let $a = \min J$ and let $b = \max J$. There are two cases according as a is even or odd.

If a is odd, then the residue is the geometry $\Gamma(C_{b+1}/C_{a-1}) \cong \Gamma((C_{a-1}^\perp \cap C_{b+1})/C_{a-1})$ of rank $b - a + 2$. We set $C_0 = \{0\}$ and $C_n = V$ for convenience.

If a is even, we may assume that $V = (C_{a-2}^\perp \cap C_{b+1})/C_{a-2}$ and that $a = 2$. Thus we need to describe the residue of C_1 . We will show that $\text{Res}_\Gamma(C_1)$ is isomorphic to a geometry $\Pi(p, H)$ defined as follows.

Definition 3.4 Note that $\dim(\text{Rad}(V)) \leq 1$. Let p be a 1-dimensional subspace of $V - \text{Rad}(V)$ and let H be some complement of p in V containing $\text{Rad}(V)$. Note that $\text{Rad}(H)$ if it is non-trivial, is not contained in p^\perp . Namely, $\text{Rad}(H) \cap p^\perp \subseteq \text{Rad}(V)$, which is 0 if $\dim(V)$ is even.

Then we define $\Pi(p, H)$ to be the geometry on the following collection of subspaces of H :

$$\{U \leq H \mid \text{Rad}(V) \not\subseteq U, \dim(\text{Rad}(U)) \leq 2 \text{ and } \text{Rad}(U) = \{0\} \text{ or } \text{Rad}(U) \not\subseteq p^\perp\}.$$

Let U and W be in $\Pi(p, H)$ with $\dim U < \dim W$. We say that U is incident to W if either $\dim(W)$ is odd and $U \subseteq W$ or $\dim(W)$ is even and $U \subseteq W - \text{Rad}(W \cap p^\perp)$.

More precisely, the objects are subspaces $U \leq H$ not containing $\text{Rad}(V)$ with the following properties

1. If U is odd dimensional then $\text{Rad}(U)$ has dimension 1 and does not lie in p^\perp .
2. If U is even dimensional then U is either non-degenerate or $\text{Rad}(U)$ has dimension 2 and is not contained in p^\perp .

Lemma 3.5 *The map given by*

$$\begin{array}{ccc} \varphi: \text{Res}_\Gamma(p) & \rightarrow & \Pi(p, H) \\ X & \mapsto & X \cap H. \end{array}$$

is an isomorphism.

Proof We first show that $X \in \text{Res}_\Gamma(p)$ if and only if $X \cap H \in \Pi(p, H)$. Note that since p is isotropic, $p^\perp \cap H$ is a codimension 1 subspace of H . Note that if $X \in \text{Res}_\Gamma(p)$, then since $X \in \Gamma$, $\text{Rad}(V) \not\subseteq X$. Moreover $X \cap H$ is a complement of p in X so $\text{Rad}(X \cap H)$ cannot have dimension more than two. Also $\text{Rad}(X \cap H) \cap p^\perp \leq \text{Rad}(X)$ and so if X is even dimensional $\text{Rad}(X \cap H) \cap p^\perp = \{0\}$ and if X is odd dimensional and $X \cap H$ is degenerate then $\text{Rad}(X \cap H) \not\subseteq p^\perp$.

Conversely if $U \in \Pi(p, H)$ it is easy to see that $X = \langle U, p \rangle$ is in $\text{Res}_\Gamma(p)$. Indeed if U is odd dimensional, since p is not perpendicular to $\text{Rad}(U)$, the space X is non-degenerate. If U is even dimensional and non-degenerate then the space X is of maximal possible rank, p is not in $\text{Rad}(X)$ and $\text{Rad}(V) \not\subseteq X$. Finally, if U is even dimensional and $\text{Rad}(U)$ has dimension 2, then, since p is not perpendicular to the whole of $\text{Rad}(U)$, we have $\text{Rad}(X) = \text{Rad}(U) \cap p^\perp \neq \text{Rad}(V)$, p and so $X \in \text{Res}_\Gamma(p)$.

Suppose that X and Y are incident elements of $\text{Res}_\Gamma(p)$ and $\dim(X) < \dim(Y)$. If $\dim(Y)$ is even then incidence is containment in both geometries. If $\dim(Y)$ is odd then we need to prove that $\text{Rad}(Y) \not\subseteq X$ iff $\text{Rad}(Y^\varphi \cap p^\perp) \not\subseteq X^\varphi$. We note that for any $Z \in \text{Res}_\Gamma(p)$, we have $\text{Rad}(Z)^\varphi = \text{Rad}(Z^\varphi \cap p^\perp)$ and so the conclusion follows. \square

Lemma 3.6 *Two points p_1, p_2 of $\Pi(p, H)$ are collinear iff $\text{Rad}(V) \not\subseteq \langle p_1, p_2 \rangle$. In particular if $\dim(V)$ is even then the collinearity graph of $\Pi(p, H)$ is a complete graph and if $\dim(V)$ is odd then the collinearity graph has diameter two.*

Proof If p_1, p_2 are two points in $\Pi(p, H)$, then $\langle p_1, p_2 \rangle$ is either totally isotropic but not contained in p^\perp or non-degenerate. So this is a line of $\Pi(p, H)$ if and only if $\text{Rad}(V) \not\subseteq \langle p_1, p_2 \rangle$. Therefore the conclusion follows. \square

Lemma 3.7 *The pre-geometry Γ is residually connected.*

Let $J \subseteq I$ and let R_J be the residue of the $(I - J)$ -flag $\{C_i\}_{i \in I - J}$. Let $\biguplus_{m=1}^M J_m$ be the partition of J into contiguous subsets. If $M \geq 2$, then the residue is connected since it is a direct product of geometries. Otherwise, the residue is isomorphic to $\Gamma(V)$ for some vector space V of dimension at least 3, or to $\Pi(C_1, H)$ inside some $\Gamma(V)$ for some vector space V of dimension at least 4. Thus the connectedness follows from Lemmas 3.3 and 3.6. \square

Corollary 3.8 *The pre-geometry Γ is a geometry with a string diagram.*

Proof By Lemma 3.2, Γ is transversal and has a string diagram and by Lemma 3.7, it is residually connected. \square

4 Simple connectedness

In this section we prove that the geometry Γ and all of its residues of rank at least 3 are simply connected. Note that Γ and all its residues are geometries with a string diagram. Therefore by Lemma 2.15 it suffices to show that all point-line cycles are null-homotopic.

Lemma 4.1 *If $|\mathbb{F}| \geq 3$ or $\dim(V)$ is even, then any point-line cycle of Γ is null-homotopic.*

Proof Let γ be a point-line cycle based at a point x . We identify γ with the sequence x_0, \dots, x_k of points on γ (so γ is in fact a $2k$ -cycle). We show by induction on k that γ is null-homotopic.

If $k \leq 3$, then $U = \langle x_0, x_1, x_2 \rangle_V$ is non-isotropic as it contains the hyperbolic line $\langle x_0, x_1 \rangle_V$. If $\text{Rad}(V) \not\subseteq U$, then it is an object of the geometry and so by Lemma 2.13, γ is null-homotopic. If $\text{Rad}(V) \subseteq U$, then since $|\mathbb{F}| \geq 3$ there is a point x such that x is collinear to x_0 , x_1 , and x_2 . Namely, consider the points $\text{Rad}(V)x_i$ in the non-degenerate space $V/\text{Rad}(V)$. These are 3 distinct points on the non-degenerate 2-space $U/\text{Rad}(V)$. Take a fourth point y on this line. Since $V/\text{Rad}(V)$ is non-degenerate, there is a point x orthogonal to y but not to any other point on $U/\text{Rad}(V)$.

Now let $k \geq 4$. If two non-consecutive points x_i and x_j in γ are collinear, then let $\gamma_1 = \delta_1 \circ \delta_2 \circ \delta_1^{-1}$ and $\gamma_2 = \delta_1 \circ \delta_3$, where

$$\begin{aligned} \delta_1 &= x_0, \dots, x_i \\ \delta_2 &= x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_i. \\ \delta_3 &= x_i, x_j, \dots, x_k. \end{aligned}$$

Clearly $\gamma_1 \circ \gamma_2$ is homotopic to γ . Also, the cycles δ_2 and γ_2 are both shorter than γ . By induction, δ_2 and γ_2 are null-homotopic and hence so are γ_1 and γ itself.

Therefore we may assume that no two non-consecutive elements in γ are collinear. In particular $x_1 \perp x_{k-1}$ and $x_{k-2} \perp x_0$. The line $\langle x_{k-1}, x_k \rangle_V$ has at least three points. Take any $y \in \langle x_{k-1}, x_k \rangle - \{x_{k-1}, x_k\}$. Then, $y \not\perp x_1, x_{k-2}$. Thus, since $x_1 \not\perp x_k = x_0$ and $x_1 \perp x_{k-1}$, the point y is collinear to x_1 . By the same reasoning y is collinear to x_{k-2} .

Now let $\gamma_1 = \delta_1 \circ \delta_2$ and $\gamma_2 = \delta_2^{-1} \circ \delta_3 \circ \delta_2$ and $\gamma_3 = \delta_2^{-1} \circ \delta_4 \circ \delta_2$, where

$$\begin{aligned}\delta_1 &= x_0, x_1, y \\ \delta_2 &= y, x_0 \\ \delta_3 &= y, x_1, x_2, \dots, x_{k-2}, y \\ \delta_4 &= y, x_{k-2}, x_{k-1}, y\end{aligned}$$

Then,

$$\gamma \simeq \gamma_1 \circ \gamma_2 \circ \gamma_3 \simeq \gamma_2 \simeq 0,$$

where the second equivalence holds since γ_1 and γ_3 are triangles, and the third equivalence holds since $\gamma_2 \simeq 0$ by induction. \square

Proposition 4.2 *The geometry Γ is simply connected.*

Proof First of all, Γ is connected by Lemma 3.3. Thus it suffices to show that any cycle is null-homotopic. By Lemma 2.15 such a cycle is homotopic to a point-line cycle. Finally, by Lemma 4.1 point-line cycles are null-homotopic and the result follows. \square

4.1 Residual geometries

We shall now prove that, apart from one exception, all residues of rank at least 3 are simply connected. As we saw above any residue is either isomorphic to $\Gamma(V)$ for some V , or to $\Pi(p, H)$ for some point p inside $\Gamma(V)$ for some V or it is a direct product of such geometries and possibly rank 1 residues.

We already proved that $\Gamma(V)$ is simply connected. We shall now prove that, apart from one exception, $\Pi(p, H)$ is a simply connected geometry. By Lemma 3.6, if n is even, then the collinearity graph of $\Pi(p, H)$ is a complete graph and so we only have to show that all triangles are null-homotopic. If n is odd, then the diameter of the collinearity graph is 2 and so all k -cycles can be decomposed into cycles of length 3, 4, and 5.

Lemma 4.3 *Suppose that $\dim(V)$ is odd. Then any cycle of length 4 or 5 can be decomposed into triangles.*

Proof We first note that two points p_1 and p_2 are at distance 2 only if $\langle p_1, p_2 \rangle \supseteq \text{Rad}(V)$. Therefore, if p_1, p_2, p_3, p_4 is a 4-cycle, then $\text{Rad}(V) \subseteq \langle p_1, p_3 \rangle, \langle p_2, p_4 \rangle$. Since $|\mathbb{F}| \geq 2$ there exists a line L on $\text{Rad}(V)$ different from these two lines and any point of $L - p^\perp$ is collinear to all of p_1, p_2, p_3, p_4 . Thus we decompose the 4-cycle into triangles. Now suppose p_1, p_2, p_3, p_4, p_5 is a 5-cycle. Then $\text{Rad}(R)$ lies on at most 1 of p_1p_3 and p_1p_4 and so one of these lines is in fact a line of the geometry. Thus, we can decompose the 5-cycle into shorter cycles. \square

Lemma 4.4 *Consider a hyperplane W of H . If $\dim(V)$ is even, then $W \in \Pi(p, H)$. If $\dim(V)$ is odd, then $W \in \Pi(p, H)$ if and only if $\text{Rad}(V) \not\subseteq W$.*

Proof Let $\dim(V)$ be even and let $S = \text{Rad}(H)$. Then $\dim(S) = 1$ and $S \not\subseteq p^\perp$. If $S \not\subseteq W$, then W is non-degenerate and so it belongs to $\Pi(p, H)$. If $S \subseteq W$, then $S \subseteq \text{Rad}(W)$, which has dimension 2 and so $\text{Rad}(W) \not\subseteq p^\perp$. So again $W \in \Pi(p, H)$.

Now let $\dim(V)$ be odd and let $R = \text{Rad}(V)$. Then $R \subseteq \text{Rad}(H)$, which has dimension 2 and is not included in p^\perp . Now since W is a hyperplane of H it either contains $\text{Rad}(H)$ or it intersects it in a 1-dimensional space. In the former case, $R \subseteq \text{Rad}(W)$ and $W \notin \Pi(p, H)$. In the latter case, either $R \subseteq \text{Rad}(W)$ and $W \notin \Pi(p, H)$, or $\text{Rad}(W) \not\subseteq p^\perp$ and $W \in \Pi(p, H)$. \square

Lemma 4.5 *Suppose $\dim(V)$ is odd. Then any triangle of $\Pi(p, H)$ is null-homotopic.*

Proof Take a triangle on the points p_1, p_2, p_3 . Note that this means that $R = \text{Rad}(V)$ does not lie on any of the lines $p_i p_j$. Let $U = \langle p_1, p_2, p_3 \rangle$. We have two cases: 1) $R \not\subseteq U$. Then pick a hyperplane W of H containing U but not R . Then by Lemma 4.4, $W \in \Pi(p, H)$. Moreover, since $\dim(W)$ is odd it is incident to each of the lines $p_i p_j$.

2) $R \subseteq U$. Let S be a point of $\text{Rad}(H) - R$. Then $S \not\subseteq p^\perp$. Also note that for any i and j , p_i, p_j, S is a triangle of type (1), so the triangle p_1, p_2, p_3 is null-homotopic. \square

Lemma 4.6 *Suppose $\dim(V) \geq 7$ or $|\mathbb{F}| \geq 3$. Then any triangle of $\Pi(p, H)$ is null-homotopic.*

Proof If $\dim(V)$ is odd, then we are done by Lemma 4.5. So now let $\dim(V) = n$ be even. Let $Q = \text{Rad}(H)$. Take a triangle on the points p_1, p_2, p_3 and let $U = \langle p_1, p_2, p_3 \rangle$. If U is a plane, then we're done by Lemma 2.13. From now on assume this is not the case. So U is either totally isotropic, or it has rank 2 and $T = \text{Rad}(U)$ is contained in $p^\perp \cap H$.

Let $L = U \cap p^\perp$. For $1 \leq i < j \leq 3$, let $L_{ij} = p_i p_j$ and $q_{ij} = L_{ij} \cap L$. Our aim is to find an object W of $\Pi(p, H)$ that is incident to all lines L_{ij} . Then, by Lemma 2.13 we're done.

Suppose we can find a point r in $p^\perp \cap H$ such that $r^\perp \supseteq L$. For such a point let $W = \langle r^\perp \cap p^\perp \cap H, U \rangle$. First note that $\dim(r^\perp \cap p^\perp \cap H) = n - 3$. This is because $p^\perp \cap H$ is a non-degenerate symplectic space of even dimension $n - 2$. Clearly $r^\perp \cap p^\perp \cap H$ has radical r .

Note that $L \subseteq r^\perp \cap p^\perp \cap H$. Also, since $U \not\subseteq p^\perp \cap H$ and $U \cap p^\perp = L$ we have $U \cap (W \cap p^\perp) = L$ and $W \cap p^\perp = r^\perp \cap p^\perp \cap H$. Hence $W = \langle (W \cap p^\perp), U \rangle$ and so $\dim(W) = \dim(W \cap p^\perp) + \dim(U) - \dim(L) = n - 2$. Thus W is an object of $\Pi(p, H)$ by Lemma 4.4 as required.

We now show that we can find such a point r that is different from q_{12} , q_{13} , and q_{23} . Then, since $r = \text{Rad}(W \cap p^\perp)$, all lines L_{ij} are incident to W as required.

There are two cases. (1) $n = 6$ and $|\mathbb{F}| \geq 3$. Since $|\mathbb{F}| \geq 3$ we can find a point $r \in L - \{q_{12}, q_{13}, q_{23}\}$. Note that L is totally isotropic and so $L \subseteq r^\perp$, as required.

(2) $n \geq 7$. Note that in this case in fact $n \geq 8$ since we assume that n is even. We find a point r in $L^\perp \cap p^\perp \cap H - L$. This is possible since $p^\perp \cap H$ is non-degenerate and L is a 2-space so that $\dim(L^\perp \cap p^\perp \cap H) = (n - 2) - 2 \geq 4 > \dim(L) = 2$. Again we have found r as required. \square

Proposition 4.7 *If $|\mathbb{F}| \geq 3$ then the residues of rank at least 3 are simply connected.*

Proof Let $J \subseteq I$ and let R_{I-J} be the residue of the J -flag $\{C_j\}_{j \in J}$. Set $r = |I - J|$. Let $\biguplus_{m=1}^M I_m$ be the partition of $I - J$ into contiguous subsets. If $M \geq 2$, then the residue is a direct product of two geometries, at least one of which is a residue of rank at least 2. Such rank 2 residues are connected by Lemma 3.7. Therefore R is simply connected in this case. Otherwise, the residue is isomorphic to $\Gamma(V)$ for some vector space V of dimension at least 3, or to $\Pi(p, H)$ for some point p inside some $\Gamma(V)$ for some vector space V of dimension at least 4. Therefore the simple connectedness follows from 4.6, 4.5 and 4.2. \square

4.2 The exceptional residue

We are now left with the intriguing case $n = 6, q = 2$. Let us first describe the geometry $\Pi(p, H)$. The points are the 16 points of $H - p^\perp$. The lines are those lines of H not contained in p^\perp . Thus there are two types of lines, totally isotropic and hyperbolic ones. Each line has exactly two points and any two points are on exactly one line.

The planes of $\Pi(p, H)$ are those non-isotropic planes of H whose radical is not contained in p^\perp . Such a plane has rank 2 and its radical is a point of $\Pi(p, H)$, i.e. it is in $H - p^\perp$. Each plane has exactly 4 points. A plane may or may not contain $\text{Rad}(H)$. Any point or line contained in a plane is incident to that plane.

The 4-spaces of $\Pi(p, H)$ are those 4-spaces of H that are either non-degenerate or have a radical of dimension 2 that is not contained in p^\perp . A 4-space W is incident

to all 8 points it contains and is incident to any line or plane it contains that doesn't pass through the point $\text{Rad}(W \cap p^\perp)$. Thus in fact W is incident to all planes Y of $\Pi(p, H)$ contained in W . Namely, if $\text{Rad}(W \cap p^\perp) \leq Y$, then $p^\perp \cap Y$ is a totally isotropic line, implying that $\text{Rad}(Y) \leq p^\perp$, a contradiction.

The geometry $\Pi(p, H)$ is not simply connected. To see this, we construct a simply connected rank 4 geometry $\overline{\Pi}(p)$ which is a degree two cover of $\Pi(p, H)$. Let us now describe the geometry $\overline{\Pi}(p)$. A point $q \in \Gamma$ is a point of $\overline{\Pi}(p)$ if the line pq is non-degenerate. In particular the points of $\Pi(p, H)$ are among the points of $\overline{\Pi}(p)$. We construct a map, which on the point set of $\overline{\Pi}(p)$ is given by

$$\begin{aligned} \psi: \overline{\Pi}(p) &\rightarrow \Pi(p, H) \\ q &\mapsto \langle p, q \rangle \cap H. \end{aligned}$$

This map is two-to-one and for any $q \in \Pi(p, H)$ we set $q^- = q$ and denote by q^+ the point of $\overline{\Pi}(p)$ such that $\psi^{-1}(q) = \{q^-, q^+\}$. We extend this notation to arbitrary point sets S of $\Pi(p, H)$ by setting $S^\epsilon = \{q^\epsilon \mid q \in S\}$ for $\epsilon = \pm$.

We now note that every object of $\Pi(p, H)$ can be identified with its point-shadow. It follows from the above description of $\Pi(p, H)$ however that inclusion of point-shadows does not always imply incidence.

In order to describe $\overline{\Pi}(p)$, we shall identify objects in $\Pi(p, H)$ and $\overline{\Pi}(p)$ with their point-shadow. We denote point-shadows by roman capitals. If we need to make the distinction between objects and their point-shadows explicit, we'll use calligraphic capitals for the objects and the related roman capitals for their point-shadows. Thus \mathcal{X} may denote an object of $\Pi(p, H)$ (or $\overline{\Pi}(p)$) whose point-shadow is X .

For any object \mathcal{X} of $\Pi(p, H)$ we will define exactly two objects \mathcal{X}_- and \mathcal{X}_+ in $\overline{\Pi}(p)$ such that $\psi(X_-) = \psi(X_+) = X$ as point-sets. We then define objects \mathcal{X} and \mathcal{Y} of $\overline{\Pi}(p)$ to be incident whenever $X \subseteq Y$ or $Y \subseteq X$ and $\psi(\mathcal{X})$ and $\psi(\mathcal{Y})$ are incident in $\Pi(p, H)$.

We shall obtain \mathcal{X}_- and \mathcal{X}_+ by defining a partition $X_0 \uplus X_1$ of the point-set of \mathcal{X} and setting $X_+ = X_0^+ \uplus X_1^-$ and $X_- = X_0^- \uplus X_1^+$.

First let \mathcal{X} be a line. If \mathcal{X} is non-degenerate, then $X_0 = X$ and $X_1 = \emptyset$ so that $X_+ = X^+$ and $X_- = X^-$. If X is isotropic, then $X = \{x_0\}$ and $X_1 = \{x_1\}$ so that $X_- = \{x_0^-, x_1^+\}$ and $X_+ = \{x_0^+, x_1^-\}$.

Next, let \mathcal{X} be a plane. Then $X_0 = \{\text{Rad}(X)\}$ and $X_1 = X - X_0$. Note that for every line \mathcal{Y} incident to \mathcal{X} the partition $Y_0 \uplus Y_1$ agrees with the partition $X_0 \uplus X_1$. Hence if \mathcal{Y} is a line and \mathcal{X} is a plane in $\overline{\Pi}(p)$ such that $\psi(\mathcal{Y})$ is incident to $\psi(\mathcal{X})$, then either $Y \subseteq X$ or $Y \cap X = \emptyset$.

Finally, let \mathcal{X} be a 4-space. Let $r = \text{Rad}(X \cap p^\perp)$. Note that the projective lines L_i ($i = 1, 2, 3, 4$) of \mathcal{X} on r meeting $H - p^\perp$ are not incident to \mathcal{X} in $\Pi(p, H)$.

First let \mathcal{X} be non-degenerate. Then L_i is non-degenerate for all i . For $i = 1, 2, 3, 4$, let $L_i = \{r, p_i, q_i\}$ such that $q_i = p_1^\perp \cap L_i$ for $i = 2, 3, 4$. Note that

this implies the fact that if $i \neq j$, then $p_i \perp q_j$ but $q_i \not\perp q_j$ and $p_i \not\perp p_j$. Then $X_0 = \{p_1, \dots, p_4\}$ and $X_1 = \{q_1, \dots, q_4\}$. We now claim that for every line \mathcal{Y} incident to \mathcal{X} the partition $Y_0 \uplus Y_1$ agrees with $X_0 \uplus X_1$. Thus we must show that if \mathcal{Y} is non-degenerate, then $Y \subseteq X_0$ or $Y \subseteq X_1$ and if \mathcal{Y} is isotropic, then Y intersects X_0 and X_1 non-trivially. First, since \mathcal{X} is non-degenerate, the lines $q_i q_j$ are all non-degenerate for $2 \leq i < j \leq 4$. Moreover, $q_1 q_i$ is non-degenerate as well, for $i = 2, 3, 4$ since otherwise L_i must be totally isotropic, a contradiction. As a consequence, $q_1 p_i$ is totally isotropic. Hence by interchanging the p_i 's for q_i 's we see that $p_i p_j$ is non-degenerate for all $1 \leq i < j \leq 4$. Considering the non-degenerate lines L_i and L_j we see that $p_i q_j$ must be isotropic for all $1 \leq i \neq j \leq 4$. We have exhausted all lines incident to \mathcal{X} and it is clear that the claim holds.

If \mathcal{X} is degenerate, then it has a radical $R = \text{Rad}(X)$ of dimension 2 passing through the radical $r = \text{Rad}(X \cap p^\perp)$. In this case the lines L_i are all totally isotropic. Let $R = L_1$. Then $X_0 = \{p_1, q_1\}$ and $X_1 = \{p_i, q_i \mid i = 2, 3, 4\}$. We now claim that for every line \mathcal{Y} incident to \mathcal{X} the partition $Y_0 \uplus Y_1$ agrees with $X_0 \uplus X_1$. Clearly every line meeting X_0 and X_1 is totally isotropic. Next consider a line \mathcal{Y} meeting L_i and L_j in X_1 . If \mathcal{Y} were totally isotropic, then $X = \langle L_1, L_i, L_j \rangle$ is totally isotropic, a contradiction. We have exhausted all lines incident to X and it is clear that the claim holds.

Next we claim that for every plane \mathcal{Y} incident to \mathcal{X} the partition $Y_0 \uplus Y_1$ agrees with $X_0 \uplus X_1$. First note that \mathcal{Y} doesn't contain $\text{Rad}(X \cap p^\perp)$, because that would make $Y \cap p^\perp$ totally isotropic and $\text{Rad}(Y) \leq p^\perp$ contrary to the description of planes of $\Pi(p, H)$. As a consequence, if \mathcal{L} is a line incident to \mathcal{Y} , then it doesn't meet $\text{Rad}(X \cap p^\perp)$ and so is incident to \mathcal{X} as well. Hence since for every line \mathcal{L} incident to \mathcal{Y} , the partition $L_0 \uplus L_1$ agrees with the partition $X_0 \uplus X_1$ as well as with the partition $Y_0 \uplus Y_1$, also the latter two partitions agree. We can conclude that if \mathcal{Y} is a point, line or plane and \mathcal{X} is a 4-space of $\overline{\Pi}(p)$ such that $\psi(\mathcal{Y})$ and $\psi(\mathcal{X})$ are incident, then either $Y \subseteq X$ or $Y \cap X = \emptyset$.

Let Ψ be the extension of ψ to the entire collection of objects in $\overline{\Pi}(p)$.

Lemma 4.8 (a) *The map $\Psi: \overline{\Pi}(p) \rightarrow \Pi(p, H)$ is 2-to-1 on the objects. For any object \mathcal{X} of $\Pi(p, H)$, if $\Psi^{-1}(\mathcal{X}) = \{\mathcal{X}_-, \mathcal{X}_+\}$, then $\psi^{-1}(X) = X_- \uplus X_+$.*

(b) *Two objects \mathcal{X} and \mathcal{Y} of $\overline{\Pi}(p)$ are incident if and only if $\Psi(\mathcal{X})$ and $\Psi(\mathcal{Y})$ are incident and $X \cap Y \neq \emptyset$.*

(c) *For any flag F_\bullet in $\overline{\Pi}(p)$, the map $\Psi: \text{Res}(F_\bullet) \rightarrow \text{Res}(\Psi(F_\bullet))$ is an isomorphism of geometries.*

(d) *The map $\Psi: \overline{\Pi}(p) \rightarrow \Pi(p, H)$ is a 2-cover.*

(e) *The pre-geometry $\overline{\Pi}(p)$ is transversal.*

Proof (a) This clear from the construction of $\overline{\Pi}(p)$.

(b) By definition of incidence in $\overline{\Pi}(p)$, \mathcal{X} and \mathcal{Y} can only be incident if $\Psi(\mathcal{X})$ and $\Psi(\mathcal{Y})$ are incident. The preceding discussion has shown that in this case either $X \cap Y = \emptyset$ or $X \subseteq Y$ or $Y \subseteq X$. Now \mathcal{X} and \mathcal{Y} are defined to be incident precisely in the latter case.

(c) Let \mathcal{X}_\bullet and \mathcal{Y}_\bullet denote objects of $\overline{\Pi}(p)$. Also, let $\mathcal{X} = \Psi(\mathcal{X}_\bullet)$, $\mathcal{Y} = \Psi(\mathcal{Y}_\bullet)$ and $F = \Psi(F_\bullet)$. By definition of the objects in $\overline{\Pi}(p)$, $\psi: X_\bullet \rightarrow X$ is a bijection of points. Therefore if \mathcal{Y} is incident with \mathcal{X} , then \mathcal{X}_\bullet is incident with exactly one object in $\Psi^{-1}(\mathcal{Y})$. Let \mathcal{Y} be incident with F . Then by the same token F_\bullet is incident with at most one object in $\Psi^{-1}(\mathcal{Y})$. We now show that there is at least one such object. Suppose that \mathcal{Y}_\bullet is incident to at least one object \mathcal{Z}_\bullet of F_\bullet . Without loss of generality assume $Y_\bullet \subseteq Z_\bullet$. If X_\bullet is an element of F_\bullet and $Z_\bullet \subseteq X_\bullet$ then \mathcal{Y}_\bullet is incident to \mathcal{X}_\bullet as well. Now assume $X_\bullet \not\subseteq Z_\bullet$. Since \mathcal{X} and \mathcal{Y} are incident, \mathcal{Y}_\bullet must be either incident to \mathcal{X}_\bullet or to the object in $\Psi^{-1}(\mathcal{X})$ different from \mathcal{X}_\bullet . In the latter case it follows that \mathcal{Z}_\bullet is incident to both objects in $\Psi^{-1}(\mathcal{X})$, a contradiction. A similar argument holds when $Z_\bullet \subseteq Y_\bullet$. We conclude that $\Psi: \text{Res}(F_\bullet) \rightarrow \text{Res}(F)$ is a bijection. Clearly incidence is preserved by Ψ , but we must show the same holds for $\Psi^{-1}: \text{Res}(F) \rightarrow \text{Res}(F_\bullet)$. Let $\mathcal{X}, \mathcal{Y} \in \text{Res}(F)$ be incident and let $\mathcal{X}_\bullet, \mathcal{Y}_\bullet \in \text{Res}(F_\bullet)$. Then there is a point q incident to \mathcal{X}, \mathcal{Y} and \mathcal{F} . Suppose $q^\epsilon \in F_\bullet$. Then $q^\epsilon \in X_\bullet \cap Y_\bullet$ and by (b) we find that \mathcal{X}_\bullet and \mathcal{Y}_\bullet are incident.

(d) This follows from (a) and (c).

(e) This is immediate from (c). □

Lemma 4.9 *The pre-geometry $\overline{\Pi}(p)$ is connected. Any two points are at distance at most 2, except the points $\text{Rad}(H)^\pm$, which are at distance 3 from one another.*

Proof Let $\epsilon \in \{+, -\}$. Let $Q = \text{Rad}(H)$. Then for any point $q \neq Q$, since the line qQ is totally isotropic, Q^ϵ is collinear to $q^{-\epsilon}$ but not to q^ϵ . In particular any two points with the same sign are at distance at most 2. It is also clear that Q^+ and Q^- have no common neighbors and are at distance at least 3.

Now consider two points $q_1, q_2 \neq Q$. If the line q_1q_2 is totally isotropic, then q_1^ϵ is collinear to $q_2^{-\epsilon}$ in $\overline{\Pi}(p)$. If the line q_1q_2 is non-degenerate, we claim that there exists $q_3 \in \Pi(p, H)$ with $q_1 \perp q_3 \not\perp q_2$. Namely, we must show that $q_1^\perp - (q_2^\perp \cup p^\perp) \neq \emptyset$. However, this is clear since both p^\perp and q_2^\perp define proper hyperplanes of the 4-space q_1^\perp . Since no linear subspace of V is the union of two of its hyperplanes our claim follows. In $\overline{\Pi}(p)$ we find both q_1^ϵ and $q_2^{-\epsilon}$ collinear to $q_3^{-\epsilon}$ so again q_1 and q_2 are at distance at most 2.

Finally consider Q^+ and Q^- . Let q_1 be a point of $\Pi(p, H)$ and let q_2 be a point of $\Pi(p, H)$ in $q_1^\perp - \{Q\}$. Then $Q^-, q_1^\perp, q_2^\perp, Q^+$ is a path of length 3. □

Lemma 4.10 *The pre-geometry $\overline{\Pi}(p)$ is residually connected.*

Proof By Lemma 4.9, $\overline{\Pi}(p)$ is connected so it suffices to show that every residue of rank at least 2 is connected. This follows immediately from Lemmas 4.8 and 3.7. \square

Lemma 4.11 *The pre-geometry $\overline{\Pi}(p)$ is a geometry with a string diagram.*

Proof By Lemma 4.8 it is transversal and by Lemma 4.10 it is residually connected. Therefore it is a geometry. That it has a string diagram is clear since it is a cover of $\Pi(p, H)$, which does have a string diagram. \square

Lemma 4.12 *The geometry $\overline{\Pi}(p)$ is simply connected.*

Proof By Lemma 4.11 and 2.15 it suffices to show that any point-line cycle is null-homotopic.

Let $Q = \text{Rad}(H)$. For any point $q \in \Pi(p, H)$, let q^* denote one of q^+ , q^- . We claim that any k -cycle with $k \geq 5$ can be decomposed into triangles, quadrangles, and pentagons. Namely, let $\gamma = q_1^*, q_2^*, \dots, q_k^*, q_1^*$ be a k -cycle in $\overline{\Pi}(p)$. If q_1 and q_4 are not both Q , then they are at distance at most 2 by Lemma 4.9 and so we can decompose γ as $(q_1^*, q_2^*, q_3^*, q_4^*) \circ \delta \circ \delta^{-1} \circ (q_4^*, \dots, q_k^*, q_1^*)$, where δ is a path from q_1^* to q_4^* of length at most 2. Thus we can decompose the k -cycle into a $(k-1)$ -cycle and a quadrangle or pentagon. If q_1 and q_2 are both Q , then replacing q_1 and q_4 by q_2 and q_5 , we can again decompose the k -cycle into a $(k-1)$ -cycle and a quadrangle or pentagon.

We shall now analyze the triangles, quadrangles, and pentagons case by case.

Triangles The points of a triangle q_1^*, q_2^*, q_3^* either all have the same sign or one has a sign different from the others. Note that a point that is collinear to another point with the same sign can not cover Q . In both cases $X = \langle q_1, q_2, q_3 \rangle$ has dimension 3. Also, X is non-isotropic because at least one of the lines is. We'll show that $r = \text{Rad}(X)$ does not lie in p^\perp . In the former case q_1, q_2, q_3 form a triangle in $\Pi(p, H)$ whose lines are non-degenerate. In particular r does not lie on any of these lines. The three remaining points on these lines are $X \cap p^\perp$. In the latter case this is because r is covered by the point of the triangle with the deviating sign. Thus X is a plane of $\Pi(p, H)$ and q_1^*, q_2^*, q_3^* belong to a plane of $\overline{\Pi}(p)$.

Quadrangles Next we consider a quadrangle $q_1^*, q_2^*, q_3^*, q_4^*$. There are four cases. Let $\epsilon \in \{+, -\}$.

(1) All points have the same sign, say ϵ . As we saw with the triangles, none of these is Q^ϵ . Hence $Q^{-\epsilon}$ is connected to all 4 points and so the quadrangle decomposes into triangles.

(2) All points but one, have the same sign, say $+$. Note that again the points on the same level do not cover Q , but are all collinear to Q^- . This decomposes the cycle into triangles and a quadrangle with two points of each sign.

(3) There are two points of each sign and these points are consecutive. Without loss of generality let $q_1^-, q_2^-, q_3^+, q_4^+$ be the quadrangle. We first note that the points q_1, q_2, q_3, q_4 are all distinct. This is because no point of $\bar{\Pi}(p)$ is collinear to both covers of the same point in $\Pi(p, H)$. Also note that q_1q_4 and q_2q_3 are totally isotropic, but q_1q_2 and q_3q_4 are not. We may also assume that q_1q_3 and q_2q_4 are non-degenerate, for otherwise we can decompose the quadrangle into triangles. Consider the space $Y = q_1^\perp \cap q_2^\perp \cap H$. It is a 3-dimensional space of rank 2 whose radical is Q . Now both $q_3^\perp \cap Y$ and $q_4^\perp \cap Y$ are lines of Y through Q . Note that $p^\perp \cap Y$ is a line of Y not through Q . Hence there is a point $q \in Y - q_3^\perp - q_4^\perp - p^\perp$. We find that q^+ is collinear to all points on the quadrangle $q_1^-, q_2^-, q_3^+, q_4^+$, which therefore decomposes into triangles.

(4) There are two points of each sign and these points are not consecutive. Without loss of generality let $q_1^-, q_2^+, q_3^-, q_4^+$ be the quadrangle. Let $X = \langle q_1, q_2, q_3, q_4 \rangle$. We first note that we can assume that the points q_1, q_2, q_3, q_4 are all distinct. No two consecutive ones can be the same so if for example $q_1 = q_3$ then the quadrangle is just a return. Hence $\dim(X) = 3, 4$. Note that if either q_1q_3 or q_2q_4 is non-degenerate, then we can decompose the quadrangle into triangles. Therefore all lines q_iq_j are totally isotropic and it follows that X is a totally isotropic 3-space. This means that $Q = q_i$ for some i , which we may assume to be 4. Consider a totally isotropic 3-space Y on q_2q_4 different from X . Then q_2q_4 and $p^\perp \cap Y$ are intersecting lines of Y and so there is a point $q \in Y - q_2q_4 - p^\perp$. Note that $q \not\in q_1, q_3$ for otherwise $\langle q, X \rangle$ is a totally isotropic 4-space. We find that q^- is collinear to all points of the quadrangle $q_1^-, q_2^+, q_3^-, q_4^+$, which therefore decomposes into triangles.

Pentagons We first note that we may assume that a pentagon has no more than 2 consecutive points of the same sign. If it contains 4 or more of sign ϵ , then these points are all collinear to $Q^{-\epsilon}$ which then yields a decomposition of the pentagon into triangles and quadrangles. If it contains exactly 3 consecutive points at the same level the same argument decomposes it into 2 triangles and a pentagon that contains no more than 2 consecutive points at the same level.

Therefore we may assume without loss of generality that the pentagon is $q_1^-, q_2^-, q_3^+, q_4^-, q_5^+$. If the point $q_4 = Q$ then we can pick q_4' to be the fourth point of $\langle q_3, q_4, q_5 \rangle - p^\perp$ and decompose the pentagon into the quadrangle $q_4'^-, q_5^+, q_4^-, q_3^+$ and the pentagon $q_1^-, q_2^-, q_3^+, q_4'^-, q_5^+$. We therefore can assume that $q_4 \neq Q$. Moreover modifying this pentagon, if necessary, by the quadrangle q_2^-, q_3^+, q_4^-, Q^+ , we may assume that $q_3 = Q$. But then q_3^+ is collinear to q_1^- as well and we can decompose

the pentagon into the triangle q_1^-, q_2^-, q_3^+ and the quadrangle $q_1^-, q_3^+, q_4^-, q_5^+$. \square

Corollary 4.13 *If $q = 2$ and $n = 6$ then the fundamental group of $\Pi(p, H)$ is $\mathbb{Z}/2\mathbb{Z}$.*

Proof The geometry $\Pi(p, H)$ has a 2-cover $\overline{\Pi}(p)$ by Lemma 4.8. This 2-cover is simply connected by Lemma 4.12. Therefore $\overline{\Pi}(p)$ is the universal cover of $\Pi(p, H)$ and the fundamental group is $\mathbb{Z}/2\mathbb{Z}$. \square

5 The group action

5.1 The classical amalgam

In this section we shall prove that $G = \text{Sp}(V)$ is the universal completion of subgroups of small rank. Our first aim is to prove Theorem 5.3 using Tits' Lemma 2.12. To this end, we proved in Section 4 that the geometry Γ is simply connected. The other result we shall need is the following.

Lemma 5.1 *The symplectic group $\text{Sp}(V)$ acts flag-transitively on Γ .*

Proof Let F_1 and F_2 be two flags of the same type. We prove the lemma by induction on $|F_1| = |F_2|$. Let M_i be the object of maximal type in F_i for $i = 1, 2$. These objects have the same isometry type since their dimensions are equal. If V is non-degenerate, then by Witt's theorem there is an isometry $g \in \text{Sp}(V)$ with $gM_1 = M_2$. Now let V be degenerate and let $R = \text{Rad}(V)$. Choose a complement V' to R containing M_1 . If $M_2 \leq V'$ as well, then again by Witt's theorem there is a $g \in \text{Sp}(V') \leq \text{Sp}(V)$ with $gM_1 = M_2$. Finally, if $M_2 \not\leq V'$, then $M_2 = M'_2 \oplus \langle v + \lambda r \rangle_V$, where $M'_2 = M_2 \cap V'$, $v \in V' - M'_2$, $\langle r \rangle = R$, and $\lambda \neq 0$. Then the transvection t that is the identity on $M'_2 \oplus R$ and maps $v \mapsto v + \lambda r$ belongs to $\text{Sp}(V)$ and satisfies $tM_2 \leq V'$. Thus, by the preceding case, there is an element $g \in \text{Sp}(V)$ sending M_1 to M_2 .

By induction, there is an element in $\text{Sp}(M_2)$ sending gF_1 to F_2 . Considering a complement V'' of R containing M_2 we see that $\text{Sp}(M_2) \leq \text{Sp}(V'')$. Clearly any element in $\text{Sp}(V'')$ can be extended to an element of $\text{Sp}(V)$ by fixing every vector in R . Thus, there is an element of $\text{Sp}(V)$ sending F_1 to F_2 , as desired. \square

Definition 5.2

Our next aim is to describe the stabilizers of flags of Γ in $G = \text{Sp}(V)$. Let $I = \{1, 2, \dots, n-1\}$. Let $C = \{C_i\}_{i \in I}$ be a chamber of Γ . For any $J \subseteq I$, let R_J be the J -residue on C and let F_J be the flag of cotype J on C . We set

$$\begin{aligned} P_J &= \text{Stab}_G(R_J) \\ B &= \text{Stab}_G(C). \end{aligned}$$

Note that $P_J = \text{Stab}_G(F_J)$ and $B = P_\emptyset = \bigcap_{J \subseteq I} P_J$. The group B is called the *Borel group* for the action of G on Γ .

Theorem 5.3 *The group $\text{Sp}(V)$ is the universal completion of the amalgam of the maximal parabolic subgroups for the action on Γ .*

Proof The group $G = \text{Sp}(V)$ acts flag-transitively on Γ by Lemma 5.1. By Proposition 4.2, Γ is simply connected and now the result follows from Tits' Lemma 2.12. \square

Corollary 5.4 *The parabolic subgroup P_J acts flag-transitively on the residue R_J .*

Proof This follows immediately from Lemma 5.1 and the definition of P_J and R_J . \square

Theorem 5.5 *Assume $|\mathbb{F}| \geq 3$. Let $J \subseteq I$ with $|J| \geq 3$. Then, the parabolic P_J of $\text{Sp}(V)$ is the universal completion of the amalgam $\{P_{J-\{j\}} \mid j \in J\}$ of rank $(r-1)$ parabolic subgroups contained in P_J .*

Proof The group P_J acts flag-transitively on the residue R_J by Corollary 5.4. By Proposition 4.7 the residue R_J is simply connected if $|J| \geq 3$. Again the result follows from Tits' Lemma 2.12. \square

Corollary 5.6 *Let $|\mathbb{F}| \geq 3$ and $\dim(V) \geq 4$. Then, $\text{Sp}(V)$ is the universal completion of the amalgam $\mathcal{A}_{\leq 2} = \{P_J \mid |J| \leq 2\}$ of rank ≤ 2 parabolic subgroups for the action on Γ .*

Proof This follows from Theorems 5.3 and 5.5 by induction on the rank. \square

The remainder of this paper is devoted to replacing even the rank ≤ 2 parabolics in the amalgamation result above by smaller groups.

5.2 Parabolic subgroups

In this section we analyze the parabolic subgroups of rank ≤ 2 of $\text{Sp}(V)$, where V is non-degenerate over a field \mathbb{F} with $|\mathbb{F}| \geq 3$. These are the groups in the amalgam $\mathcal{A}_{\leq 2}$ of Corollary 5.6. In order to study these groups in some detail we will use the following setup. Let $n = 2r$. and $\mathcal{H} = \{e_i, f_j \mid 1 \leq i \leq r, 1 \leq j \leq r\}$ be a hyperbolic basis corresponding to C as in Corollary 3.1. That is, if we relabel \mathcal{H} such that

$$\begin{aligned} h_{2i-1} &= e_i & \text{for } 1 \leq i \leq r, \\ h_{2j} &= f_j & \text{for } 1 \leq j \leq r, \end{aligned}$$

then, $C = \{C_l\}_{l \in I}$, where $C_l = \langle h_1, h_2, \dots, h_l \rangle_V$. For $i = 1, 2, \dots, r$, let $H_i = \langle e_i, f_i \rangle_V$. We will use this setup throughout the remainder of the paper.

Let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix defining the symplectic form with respect to the basis \mathcal{H} is

$$S = \begin{pmatrix} E & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & E \end{pmatrix}$$

Since the Borel group is contained in all parabolic subgroups, even the ones of rank ≤ 2 , we must know exactly what it looks like.

Lemma 5.7 *We have*

$$B \cong (\mathbb{F} \rtimes \mathrm{GL}_1(\mathbb{F}))^{(n/2)}$$

where, for $j = 1, 2, \dots, r$, $\mathbb{F} \rtimes \mathrm{GL}_1(\mathbb{F})$ is realized on $\langle e_j, f_j \rangle$ as

$$B_j = \left\{ \begin{pmatrix} a_j & b_j \\ 0 & a_j^{-1} \end{pmatrix} \mid a_j \in \mathbb{F}^*, b_j \in \mathbb{F} \right\}.$$

Also, the kernel of the action of G on Γ is $H = \{\pm 1\}$

Proof First note that if $n = 2$, then $G = \mathrm{SL}_2(\mathbb{F})$ and the stabilizer of C is the usual Borel group

$$B_1 = \left\{ \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix} \mid a_1 \in \mathbb{F}^*, b_1 \in \mathbb{F} \right\}.$$

Clearly, $B_1 \cong \mathbb{F} \rtimes \mathrm{GL}_1(\mathbb{F})$.

Now let $n \geq 3$. We observe that, if $g \in G$ stabilizes C , then it also stabilizes $C_l^\perp \cap C_m$ for any $1 \leq l \leq m \leq n-1$. Hence, g stabilizes the subspaces spanned by e_i and it stabilizes the subspaces spanned by $\{e_i, f_i\}$ for all $1 \leq i \leq r$. Thus

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_r \end{pmatrix}$$

where if $j = 1, 2, \dots, r$ we have

$$g_j = \begin{pmatrix} a_j & b_j \\ 0 & a_j^{-1} \end{pmatrix}$$

for some $a_j \in \mathbb{F}^*$ and $b_j \in \mathbb{F}$.

The kernel of the action is given by $\bigcap_{x \in G} xBx^{-1}$. This means that if g is described as above, then, for all $1 \leq j \leq r$ we have $b_j = 0$ and $a_j = a_j^{-1} = a$ for some fixed $a \in \mathbb{F}^*$. This means $a = \pm 1$. The result follows. \square

We now focus on the parabolic subgroups of rank ≤ 2 . Since we want to make a distinction between the various rank ≤ 2 parabolic subgroups, we shall give them an individual name.

Definition 5.8 We assign the following names to the various rank ≤ 2 parabolic subgroups:

$$\begin{aligned} S_j &= P_{2j-1} && \text{for } 1 \leq j \leq r \\ M_i &= P_{2i} && \text{for } 1 \leq i \leq r-1 \\ S_{ij} &= P_{2i-1, 2j-1} && \text{for } 1 \leq i < j \leq r \\ M_{ij} &= P_{2i, 2j} && \text{for } 1 \leq i < j \leq r-1 \\ Q_{ij} &= P_{2i, 2j-1} && \text{for } 1 \leq i \leq r-1 \text{ and } 1 \leq j \leq r \end{aligned}$$

Thus the collection of groups in the amalgam $\mathcal{A}_{\leq 2}$ is $\{M_i, S_j, S_{jl}, M_{ik}, Q_{ij} \mid 1 \leq i, k \leq r-1 \text{ and } 1 \leq j, l \leq r\}$.

Definition 5.9 In order to describe the groups in $\mathcal{A}_{\leq 2}$ abstractly and as matrix groups we define the following matrix groups:

$$M = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & w \\ 0 & a_1^{-1} & 0 & 0 \\ 0 & a_1^{-1}wa_2 & a_2 & b_2 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix} \mid a_1, a_2 \in \mathbb{F}^*, b_1, b_2, w \in \mathbb{F} \right\},$$

$$S = \text{SL}_2(\mathbb{F}) \cong \text{Sp}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}, \text{ where } ad - bc = 1 \right\},$$

$$\begin{aligned} M_* &= \left\{ \left(\begin{array}{cc|cc|cc} a_1 & b_1 & 0 & w_1 & 0 & 0 \\ 0 & a_1^{-1} & 0 & 0 & 0 & 0 \\ \hline 0 & a_1^{-1}w_1a_2 & a_2 & b_2 & 0 & w_3 \\ 0 & 0 & 0 & a_2^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & a_2^{-1}w_3a_3 & a_3 & b_3 \\ 0 & 0 & 0 & 0 & 0 & a_3^{-1} \end{array} \right) \mid \begin{array}{l} a_1, a_2, a_3 \in \mathbb{F}^* \\ b_1, b_2, b_3, w_1, w_3 \in \mathbb{F} \end{array} \right\}, \\ Q_- &= \left\{ \left(\begin{pmatrix} a_1 & b_1 & 0 & w_1 \\ c_1 & d_1 & 0 & w_2 \\ v_2 & v_1 & a_2 & b_2 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, a_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1d_1 - b_1c_1 = 1 \\ a_1w_2 - c_1w_1 + v_2a_2^{-1} = 0 \\ b_1w_2 - d_1w_1 + v_1a_2^{-1} = 0 \end{array} \right) \right\}, \end{aligned}$$

and

$$Q_+ = \left\{ \begin{pmatrix} a_2 & b_2 & v_2 & v_1 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & w_1 & a_1 & b_1 \\ 0 & w_2 & c_1 & d_1 \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, a_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 = 1 \\ a_1 w_2 - c_1 w_1 + v_2 a_2^{-1} = 0 \\ b_1 w_2 - d_1 w_1 + v_1 a_2^{-1} = 0 \end{array} \right\}.$$

Lemma 5.10 (a) *For all indices that apply, we have the following isomorphisms:*

$$\begin{aligned} S_j &\cong S \times \Pi_{i \neq j} B_i, \\ M_i &\cong M \times \Pi_{j \neq i, i+1} B_j, \\ S_{ij} &= \langle S_i, S_j \rangle_G, \\ M_{ij} &= \langle M_i, M_j \rangle_G, \\ Q_{ij} &= \langle M_i, S_j \rangle_G. \end{aligned}$$

Moreover, we have the following isomorphisms,

(b)

$$M_{ij} \cong \begin{cases} M_* \times \Pi_{k \neq i, i+1, j, j+1} B_k & \text{if } |i - j| = 1 \\ M \times M \times \Pi_{k \neq i, i+1, j, j+1} B_k & \text{if } |i - j| \geq 2 \end{cases},$$

(c)

$$S_{ij} \cong (S \times S) \times \Pi_{k \neq i, j} B_k,$$

(d) *If $j \notin \{i, i+1\}$, then*

$$Q_{ij} \cong (M \times S) \times \Pi_{k \neq i, i+1, j} B_k.$$

Furthermore,

$$Q_{ii} \cong Q_- \times \Pi_{k \neq i, i+1} B_k,$$

and

$$Q_{ii+1} \cong Q_+ \times \Pi_{k \neq i, i+1} B_k.$$

Proof (a) By definition S_i is the stabilizer of the flag of type $I - \{2i - 1\}$ on the standard chamber C . Therefore it acts on H_j as B_j for all $j \neq i$ and it stabilizes and acts as $S \cong \mathrm{Sp}_2(\mathbb{F})$ on H_i . Similarly, by definition, M_i acts on H_j as B_j for all $j \neq i, i+1$ and it stabilizes $\langle e_i \rangle$ and $\langle e_i, f_i, e_{i+1} \rangle$ and $H_i \oplus H_{i+1}$. Therefore it acts as M on $H_i \oplus H_{i+1}$. The last three equalities follow from the fact that rank 2 residues are connected.

The remaining isomorphisms follow from similar considerations. The parabolic subgroup under consideration acts as B_j on H_j for all but 2, 3, or 4 values of j . Then the action on the subspace generated by the remaining H_j determines the non-Borel part (S, M, M_*, Q_-, Q_+) of the group. \square

6 The slim amalgam

In this section we define a slim version \mathcal{A}^π of the amalgam $\mathcal{A}_{\leq 2}$ by eliminating a large part of the Borel group from each of its groups. More precisely, the collection of groups in \mathcal{A}^π is $\{X^\pi \mid X \in \mathcal{A}_{\leq 2}\}$, where X^π is given in Definitions 6.1 and 6.3. Note that the groups X^π are subgroups of G and the inclusion maps for the amalgam \mathcal{A}^π are the inclusion maps induced by G . In Section 7 we shall construct an abstract version of \mathcal{A}^π , whose groups are not considered to be subgroups of G .

Definition 6.1 For $1 \leq i \leq r-1$, the group M_i^π fixes every vector in H_k , for every $k \neq i, i+1$ and a generic element acts on $H_i \oplus H_{i+1}$ as

$$m(b_1, w, b_2) = \begin{pmatrix} 1 & b_1 & 0 & w \\ 0 & 1 & 0 & 0 \\ 0 & w & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where } w, b_1, b_2 \in \mathbb{F}.$$

For $1 \leq j \leq r$, the group S_j^π fixes every vector in H_k , for every $k \neq j$ and acts on H_j as $\text{Sp}(H_j) \cong \text{Sp}_2(\mathbb{F})$. A generic element of S_j^π is denoted

$$s(a, b, c, d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the matrix defines the action on H_j with respect to the basis $\{e_j, f_j\}$.

Note 6.2 We note that S_j^π is generated by two opposite long-root groups.

Definition 6.3 For $1 \leq i < j \leq r$, let

$$S_{ij}^\pi = \langle S_i^\pi, S_j^\pi \rangle_G.$$

For $1 \leq i < j \leq r-1$, let

$$M_{ij}^\pi = \langle M_i^\pi, M_j^\pi \rangle_G.$$

For $1 \leq i \leq r-1$ and $1 \leq j \leq r$, let

$$Q_{ij}^\pi = \langle M_i^\pi, S_j^\pi \rangle_G.$$

Definition 6.4 For any $i = 1, 2, \dots, r$, we set

$$U_i^\pi = \begin{cases} M_i^\pi \cap S_i^\pi & \text{if } 1 \leq i < r \\ M_{i-1}^\pi \cap S_i^\pi & \text{if } i = r \end{cases},$$

and

$$B_i^\pi = N_{S_i^\pi}(U_i^\pi).$$

Moreover,

$$B^\pi = \langle B_1^\pi, \dots, B_r^\pi \rangle \cong B_1^\pi \times \dots \times B_r^\pi.$$

Definition 6.5 In order to describe the groups in \mathcal{A}^π abstractly and as matrix groups we define the following matrix groups:

$$M^\pi = \left\{ \begin{pmatrix} 1 & b_1 & 0 & w \\ 0 & 1 & 0 & 0 \\ 0 & w & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid b_1, b_2, w \in \mathbb{F} \right\},$$

$$S^\pi = S,$$

$$M_*^\pi = \left\{ \left(\begin{array}{cc|cc|cc} 1 & b_1 & 0 & w_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & w_1 & 1 & b_2 & 0 & w_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & w_3 & 1 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid b_1, b_2, b_3, w_1, w_3 \in \mathbb{F} \right\},$$

$$Q_-^\pi = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & w_1 \\ c_1 & d_1 & 0 & w_2 \\ v_2 & v_1 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 = 1 \\ v_2 = -a_1 w_2 + c_1 w_1 \\ v_1 = -b_1 w_2 + d_1 w_1 \end{array} \right\},$$

and

$$Q_+^\pi = \left\{ \begin{pmatrix} 1 & b_2 & v_2 & v_1 \\ 0 & 1 & 0 & 0 \\ 0 & w_1 & a_1 & b_1 \\ 0 & w_2 & c_1 & d_1 \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 = 1 \\ v_2 = -a_1 w_2 + c_1 w_1 \\ v_1 = -b_1 w_2 + d_1 w_1 \end{array} \right\}.$$

Lemma 6.6 For $i = 1, 2, \dots, r-1$ and $j = 1, 2, \dots, r$, the sets M_i^π and S_j^π are subgroups of $\text{Sp}(V)$ and

- (a) $M_i^\pi \cong M^\pi \cong \mathbb{F}^3$,
- (b) $S_j^\pi \cong S^\pi \cong \text{Sp}_2(\mathbb{F})$.

Proof Setting $m = m(b_1, w, b_2)$ one verifies easily that (me_i, mf_i) and (me_{i+1}, mf_{i+1}) are two orthogonal hyperbolic pairs. Hence m is a symplectic matrix. Clearly $S_j^\pi = \text{Stab}_{\text{Sp}(V)}(H_j)$ is a subgroup of $\text{Sp}(V)$.

- (a) It is straightforward to check that M_i^π is an abelian group isomorphic to \mathbb{F}^3 .
- (b) This is true by definition. \square

Lemma 6.7 *For all indices $i \neq j$ that apply, we have*

(a)

$$M_{ij}^\pi \cong \begin{cases} M_*^\pi & \text{if } |i - j| = 1 \\ M_i^\pi \times M_j^\pi & \text{if } |i - j| \geq 2 \end{cases},$$

(b)

$$S_{ij}^\pi \cong S_i^\pi \times S_j^\pi.$$

(c) *If $j \notin \{i, i + 1\}$, then*

$$Q_{ij}^\pi \cong M_i^\pi \times S_j^\pi.$$

Furthermore,

$$\begin{aligned} Q_{ii}^\pi &\cong Q_-^\pi, \\ Q_{ii+1}^\pi &\cong Q_+^\pi. \end{aligned}$$

Proof Part (a) and (b) are straightforward. As for part (c), if $j \notin \{i, i + 1\}$, then clearly M_i^π and S_j^π commute and intersect trivially.

We now turn to the cases Q_{ii}^π and Q_{ii+1}^π . First note that conjugation by the permutation matrix that switches (e_i, f_i) and (e_{i+1}, f_{i+1}) , interchanges S_i^π and S_{i+1}^π , but fixes every element in M_i^π . Thus it suffices to prove the claim for Q_{ii}^π .

We consider Q_-^π to be represented as a matrix group with respect to the basis $\{e_i, f_i, e_{i+1}, f_{i+1}\}$. We shall now prove that with this identification $Q_-^\pi = Q_{ii}^\pi$. To this end we show that Q_-^π is the stabilizer of the vector e_{i+1} in $\text{Sp}(H_i \oplus H_{i+1})$. It is clear from the shape of the third column that Q_-^π stabilizes e_{i+1} . On the other hand, any matrix A in $\text{Sp}(H_i \oplus H_{i+1})$ stabilizing e_{i+1} must have such a third column. It must also have zeroes in the last row as in Q_-^π since in fixing e_{i+1} it must also stabilize e_{i+1}^\perp . Any such matrix A must satisfy the conditions on the entries as indicated in the description of Q_-^π since $\{Ae_i, Af_i, e_{i+1}, Af_{i+1}\}$ must be isometric to $\{e_i, f_i, e_{i+1}, f_{i+1}\}$ in order for A to be symplectic. Therefore Q_-^π equals this stabilizer and hence is a group.

Clearly Q_-^π contains S_i^π and M_i^π . We now show that $\langle S_i^\pi, M_i^\pi \rangle = Q_-^\pi$. We note that if $m = m(0, w, b_2)$ and $s = s(a, b, c, d)$, $s' = s(a_1, b_1, c_1, d_1)$ then $sms^{-1}s'$ is the

following matrix:

$$\begin{pmatrix} a_1 & b_1 & 0 & aw \\ c_1 & d_1 & 0 & cw \\ v_2 & v_1 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where v_1, v_2 are as in the definition of Q_-^π . Therefore all the elements of Q_-^π can be obtained that way. □

Lemma 6.8 *We have*

(a)

$$M_{ij}^\pi \cong \begin{cases} \mathbb{F}^5 & \text{if } |i - j| = 1 \\ \mathbb{F}^6 & \text{if } |i - j| \geq 2 \end{cases},$$

(b)

$$S_{ij}^\pi \cong \mathrm{Sp}_2(\mathbb{F}) \times \mathrm{Sp}_2(\mathbb{F}),$$

(c) *If $j \notin \{i, i + 1\}$, then*

$$Q_{ij}^\pi \cong \mathbb{F}^3 \times \mathrm{Sp}_2(\mathbb{F}).$$

Furthermore,

$$Q_{ii}^\pi \cong (U \times V) \rtimes S_i^\pi \cong \mathbb{F}^3 \rtimes \mathrm{Sp}_2(\mathbb{F}).$$

Here, taking the labelling as in Lemma 6.7 we have

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid b_2 \in \mathbb{F} \right\} \cong \mathbb{F},$$

$$V = \left\{ \begin{pmatrix} 1 & 0 & 0 & w_1 \\ 0 & 1 & 0 & w_2 \\ v_2 & v_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{ll} w_1, w_2, v_1, v_2 \in \mathbb{F} \text{ such that} \\ v_2 & = -w_2 \\ v_1 & = w_1 \end{array} \right\} \cong \mathbb{F}^2,$$

$$S_i^\pi = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{ll} a_1, b_1, c_1, d_1 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 & = 1 \end{array} \right\} \cong \mathrm{Sp}_2(\mathbb{F}).$$

Moreover, the action of S_i^π on V by conjugation is the natural action from the left of $\mathrm{Sp}_2(\mathbb{F})$ on V . Also $U = Z(Q_{ii}^\pi)$.

Finally,

$$Q_{ii+1}^\pi \cong Q_{ii}^\pi$$

where the isomorphism is given by the labelling of the entries.

Proof (a) and (b): This is straightforward. As for (c), if $j \notin \{i, i+1\}$, then the result follows from the corresponding part in Lemma 6.7.

Since $M_i^\pi \leq V \leq Q_{ii}^\pi$ and $U \leq Q_{ii}^\pi$, we have $Q_{ii}^\pi = \langle M_i^\pi, S_i^\pi \rangle = \langle V, S_i^\pi \rangle = \langle U, V, S_i^\pi \rangle$. Also, $U \cap V = 1$ and $\langle U, V \rangle$ is abelian so that $\langle U, V \rangle \cong U \times V$. Furthermore, $\langle U, V \rangle \cap S_i^\pi = 1$ and $S_i^\pi \leq N_{Q_{ii}^\pi}(\langle U, V \rangle)$. In fact $S_i^\pi \leq C_{Q_{ii}^\pi}(U)$ and one calculates that conjugation of V by S_i^π is the natural action from the left of $\mathrm{Sp}_2(\mathbb{F})$ on $\mathbb{F}^2 = V$.

We now show that $U = Z(Q_{ii}^\pi)$. This follows from a general argument about semidirect products using the fact that S_i^π acts faithfully and fixed point free on V .

The isomorphism $Q_{ii}^\pi \cong Q_{ii+1}^\pi$ is given by conjugation as in the proof of Lemma 6.7.

□

7 The concrete amalgam

Definition 7.1 Let J be some index set. A *concrete amalgam* of finite rank $s \in \mathbb{Z}_{\geq 0}$ over J is a collection $\mathcal{A} = \{A_K \mid K \subseteq J, |K| \leq s\}$ of groups together with inclusion homomorphisms $\varphi_{M,K}: A_K \rightarrow A_M$ for every pair (M, K) with $M \subseteq K$ satisfying $\varphi_{L,M} \circ \varphi_{M,K} = \varphi_{L,K}$ whenever $L \subseteq M \subseteq K$.

The *universal completion* of \mathcal{A} is then a group \widehat{G} whose elements are words in the elements of the groups in \mathcal{A} subject to the relations between the elements of A_K for any $K \subseteq J$ and in which for each $K \subseteq M \subseteq J$ each $a \in A_K$ is identified with $\varphi_{M,K}(a) \in A_M$.

Let $\widehat{\cdot}: \mathcal{A} \rightarrow \widehat{G}$ be the embedding of \mathcal{A} into \widehat{G} .

Note 7.2

- (i) For each $K \subseteq J$ with $|K| \leq s$, $\widehat{\cdot}: A_K \rightarrow \widehat{A_K} \leq \widehat{G}$ is a homomorphism, which is surjective, but not necessarily injective.
- (ii) We have $\widehat{A_K} \leq \widehat{A_M}$ whenever $K \subseteq M$.
- (iii) Although it may be the case in some situations, we do not assume that $\widehat{A_{K \cap M}} = \widehat{A_K} \cap \widehat{A_M}$.

Definition 7.3 We will define a concrete amalgam \mathcal{A}° . Its set of subgroups is $\{M_{ij}^\circ, Q_{ik}^\circ, S_{kl}^\circ, M_i^\circ, S_k^\circ \mid 1 \leq i, j \leq r-1, 1 \leq k, l \leq r\}$, where for each $X \in \mathcal{A}_{\leq 2}$, X° is a copy of X^π and the inclusion homomorphisms are as follows:

$$\begin{aligned} \varphi_{i,\{i,j\}}^P &: M_i^\circ \rightarrow M_{ij}^\circ \\ \varphi_{j,\{i,j\}}^P &: M_j^\circ \rightarrow M_{ij}^\circ \\ \varphi_{k,\{k,l\}}^S &: S_k^\circ \rightarrow S_{kl}^\circ \\ \varphi_{l,\{k,l\}}^S &: S_l^\circ \rightarrow S_{kl}^\circ \\ \varphi_{i,\{i,k\}}^{PQ} &: M_i^\circ \rightarrow Q_{ik}^\circ \\ \varphi_{k,\{i,k\}}^{SQ} &: S_k^\circ \rightarrow Q_{ik}^\circ. \end{aligned}$$

These inclusions are given by the presentations of X^π as matrix groups as given in Definitions 6.1 and 6.3. We denote the universal completion of \mathcal{A}° by G° .

Definition 7.4 We now define the map $\pi: \mathcal{A}^\circ \rightarrow G$. For any $X \in \mathcal{A}_{\leq 2}$, it identifies X° with its isomorphic copy, X^π , in G . Thus the image of \mathcal{A}° under π is \mathcal{A}^π .

Lemma 7.5 *For any element $X \in \mathcal{A}_{\leq 2}$, the map $\pi: X^\circ \rightarrow X^\pi$ is an isomorphism.*

Proof This is true by the definition of π . \square

It then follows that the map π extends to a surjective map from the universal cover G° to G .

Lemma 7.6 *If $i \geq 2$ then $U_i^\circ = S_i^\circ \cap M_i^\circ = S_i^\circ \cap M_{i-1}^\circ = M_i^\circ \cap M_{i-1}^\circ$.*

Proof Note that the above are true for the images under π . By definition the group M_i^π stabilizes the spaces H_k if $k \neq i, i+1$ and it also stabilizes e_i and e_{i+1} . Moreover S_i^π stabilizes all the H_k for $k \neq i$. Therefore $S_i^\pi \cap M_i^\pi$ stabilizes all vectors in H_k for $k \neq i$ and it also stabilizes e_i . Similarly for $S_i^\pi \cap M_{i-1}^\pi$ and $M_i^\pi \cap M_{i-1}^\pi$. We notice that π is an isomorphism when restricted to \mathcal{A}° and so the conclusion follows. \square

Definition 7.7 For $i = 1, 2, \dots, r$, we define the following subgroup of $S_i^\circ \in \mathcal{A}^\circ$: $B_i^\circ = N_{S_i^\circ}(U_i^\circ)$. Furthermore, we set $B^\circ = \langle B_i^\circ \mid i = 1, 2, \dots, r \rangle$ as a subgroup of B° .

Lemma 7.8 *For $i = 1, 2, \dots, r$, we have*

$$(a) \ U_i^\pi = \pi(U_i^\circ) \text{ and } B_i^\pi = \pi(B_i^\circ).$$

(b)

$$U_i^\circ = \left\{ \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \mid b_1 \in \mathbb{F} \right\} \cong \mathbb{F}.$$

(c)

$$B_i^\circ = \left\{ \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \mid a_1, b_1, d_1 \in \mathbb{F} \text{ such that } a_1 d_1 = 1 \right\} \cong \mathbb{F} \rtimes \mathbb{F}^*.$$

Lemma 7.9 *For $X \in \mathcal{A}_{\leq 2}$, B° normalizes X° . The action is described by the following*

1. $[B_i^\circ, S_j^\circ] = 1$ if $i \neq j$ and B_i° acts on S_i° as inner automorphisms.
2. $[B_i^\circ, M_j^\circ] = 1$ if $j \neq i, i+1$. B_i° acts on M_i° as the conjugation in Q_{ii}° and B_{i+1}° acts on M_i° as the conjugation in Q_{ii+1}° .
3. The action on the rank 2 parabolics is determined by the action above on the rank one parabolics.

Proof Note that the actions are as above for the B_i^π acting on the various X^π . Moreover the map π is an isomorphism when restricted to the various $X^\circ \in \mathcal{A}^\circ$. Since $B_i^\circ, S_j^\circ \leq S_{ij}^\circ$, the action of B_i° on S_j° is the same as the action of B_i^π on S_j^π . For part 2 we note that $B_i^\circ, M_j^\circ \leq Q_{ij}^\circ$ and so the action of B_i° on M_j° corresponds to the action of B_i^π on M_j^π .

Part 3 follows immediately since the groups in \mathcal{A}° are all embedded in G° . \square

Lemma 7.10 (a) *The group B° is the internal direct product*

$$B^\circ = B_1^\circ \times B_2^\circ \times \cdots \times B_r^\circ,$$

(b)

$$\pi: B^\circ \rightarrow B^\pi \text{ is an isomorphism.}$$

Proof For any $1 \leq i < j \leq r$, we have $B_i^\circ \leq S_i^\circ$ and since $S_{ij}^\circ = S_i^\circ \times S_j^\circ$ we have $\langle B_1^\circ, B_j^\circ \rangle = B_i^\circ \times B_j^\circ$. \square

Lemma 7.11 *For $X \in \mathcal{A}_{\leq 2}$ we have $\langle B^\pi, X^\pi \rangle_G = X$.*

Proof This is an easy calculation inside G . \square

Lemma 7.12 *For any $X \in \mathcal{A}_{\leq 2}$, $\pi(X^\circ \cap B^\circ) = X^\pi \cap B^\pi$.*

Proof The inclusion \subseteq is trivial. We now prove \supseteq . We do this case by case for any $X \in \mathcal{A}$.

Let $X = S_i$. Then $X^\pi \cap B^\pi = B_i^\pi$ and since $B_i^\circ \leq S_i^\circ \cap B^\circ$, we find $\pi(X^\circ \cap B^\circ) \supseteq \pi(B_i^\circ) = B_i^\pi$.

Let $X = P_i$. Then $X^\pi \cap B^\pi = U_i^\pi \times U_{i+1}^\pi$ and since $U_i^\circ \times U_{i+1}^\circ \leq M_i^\circ \cap B^\circ$, we find $\pi(X^\circ \cap B^\circ) \supseteq \pi(U_i^\circ \times U_{i+1}^\circ) = U_i^\pi \times U_{i+1}^\pi$.

Let $X = S_{ij}$. Then $X^\pi \cap B^\pi = B_i^\pi \times B_j^\pi$ and since $B_i^\circ \times B_j^\circ \leq S_{ij}^\circ \cap B^\circ$, we find $\pi(X^\circ \cap B^\circ) \supseteq \pi(B_i^\circ \times B_j^\circ) = B_i^\pi \times B_j^\pi$.

Let $X = Q_{ij}$. Then $X^\pi \cap B^\pi = \langle U_i^\pi, U_{i+1}^\pi, B_j^\pi \rangle$ and since $\langle U_i^\circ, U_{i+1}^\circ, B_j^\circ \rangle \leq Q_{ij}^\circ \cap B^\circ$, we find $\pi(X^\circ \cap B^\circ) \supseteq \pi(\langle U_i^\circ, U_{i+1}^\circ, B_j^\circ \rangle) = \langle U_i^\pi, U_{i+1}^\pi, B_j^\pi \rangle$.

Let $X = P_{ij}$. Then $X^\pi \cap B^\pi = \langle U_i^\pi, U_{i+1}^\pi, U_j^\pi, U_{j+1}^\pi \rangle$ and since $\langle U_i^\circ, U_{i+1}^\circ, U_j^\circ, U_{j+1}^\circ \rangle \leq M_{ij}^\circ \cap B^\circ$, we find $\pi(X^\circ \cap B^\circ) \supseteq \pi(\langle U_i^\circ, U_{i+1}^\circ, U_j^\circ, U_{j+1}^\circ \rangle) = \langle U_i^\pi, U_{i+1}^\pi, U_j^\pi, U_{j+1}^\pi \rangle$. \square

Our next aim is to define a map $\chi: \mathcal{A}_{\leq 2} \rightarrow G^\circ$ extending $\pi^{-1}: X^\pi \rightarrow X^\circ$ for every $X \in \mathcal{A}_{\leq 2}$.

Lemma 7.13 χ is well-defined on $X^\pi \cap B^\pi$ for all $X \in \mathcal{A}_{\leq 2}$.

Proof This follows from Lemma 7.12 \square

Define χ on $X = B^\pi X^\pi$ for any $X \in \mathcal{A}_{\leq 2}$ as follows: $\chi(bx) = \chi(b)\chi(x)$.

Lemma 7.14 χ is well-defined and injective on X .

Proof Note that if $b, b' \in B^\pi$ and $x, x' \in X^\pi$ then $bx = b'x' \Rightarrow b^{-1}b' = xx'^{-1} \in B^\pi \cap X^\pi$. Moreover by Lemma 7.13 $\chi(b^{-1}b') = \chi(xx'^{-1})$ and so $\chi(bx) = \chi(b'x')$ and χ is well defined.

Also if $bx \in X$ with $\chi(bx) = 1$ it follows that $\chi(b) = \chi(x)^{-1} \in B^\circ \cap X^\circ$ (because χ is the inverse of π when restricted to B^π and X^π). Therefore $b = \pi(\chi(b))$ and $x = \pi(\chi(x))$ are both in $\pi(X^\circ \cap B^\circ) = X^\pi \cap B^\pi$. But χ is a bijection when restricted to $X^\pi \cap B^\pi$ and so $b = x^{-1}$ and χ is injective. \square

Lemma 7.15 χ is an embedding of the amalgam $\mathcal{A}_{\leq 2}$ into G° .

Proof Let $X \in \mathcal{A}$ then by lemma 7.11 and Lemma 7.9 we know that $X = B^\pi X^\pi$ and so if $bx, b'x' \in X$ then $\chi(b'x'bx) = \chi(b'b)\chi(b^{-1}x'bx) = \chi(b'b)\chi(b^{-1}x'b)\chi(x)$. Also $\chi(b'x')\chi(bx) = \chi(b')\chi(x')\chi(b)\chi(x) = \chi(b')\chi(b)\chi(b^{-1})\chi(x')\chi(b)\chi(x)$. so we only need to prove that $\chi(b^{-1})\chi(x')\chi(b) = \chi(b^{-1}x'b)$ which is equivalent to the fact that the action of B^π on X^π is the same as the action of B° on X° . This follows from Lemma 7.9. \square

As a consequence, we find the following result.

Proposition 7.16 The map χ extends to a surjective homomorphism $G \rightarrow G^\circ$ which we also denote by χ .

Proof By Lemma 7.15 the map χ extends to a homomorphism $G \rightarrow G^\circ$ whose image contains the subgroup of G° generated by the subgroups in \mathcal{A}° . Since $\langle \mathcal{A}^\circ \rangle = G^\circ$, the conclusion follows. \square

Proof Of Theorem 1.1. By Corollary 5.6 G is the universal completion of the amalgam $\mathcal{A}_{\leq 2}$.

The map $\pi: \mathcal{A}^\circ \rightarrow G$ given by $\pi: X^{c\text{irc}} \rightarrow X^\pi$ for all $X \in \mathcal{A}_{\leq 2}$ extends to surjection $\pi: G^\circ \rightarrow H^\pi$ where H^π is the subgroup of G generated by the subgroups in \mathcal{A}^π . By Lemma 7.11 $H^\pi = G$, and so $\pi: G^\circ \rightarrow G$ is surjective.

By Proposition 7.16 there is a surjective map $\chi: G \rightarrow G^\circ$, which by Lemma 7.14 is injective on the subgroups of $\mathcal{A}_{\leq 2}$.

The composition $\chi \circ \pi: G^\circ \rightarrow G^\circ$ is a surjective homomorphism which is the identity on every subgroup in the amalgam \mathcal{A}° . Since G° is the universal completion of \mathcal{A}° , the only such map is the identity.

Hence $\pi: G^\circ \rightarrow G$ is an isomorphism. \square

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